Lect 6

Image Theory.

Arrays of two isotropic point sources.

Image Theory

Vertical Electric Dipole (VED)

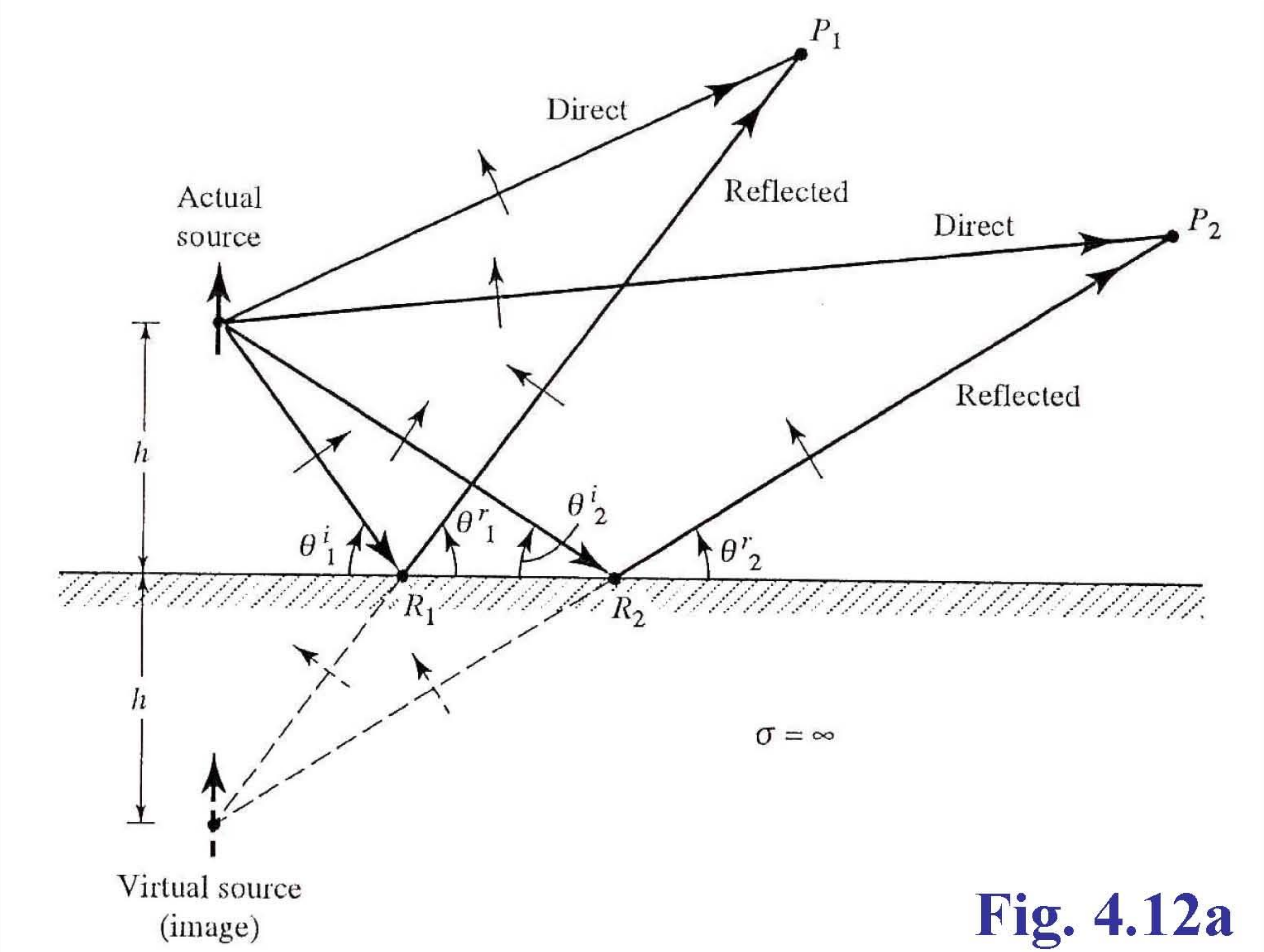
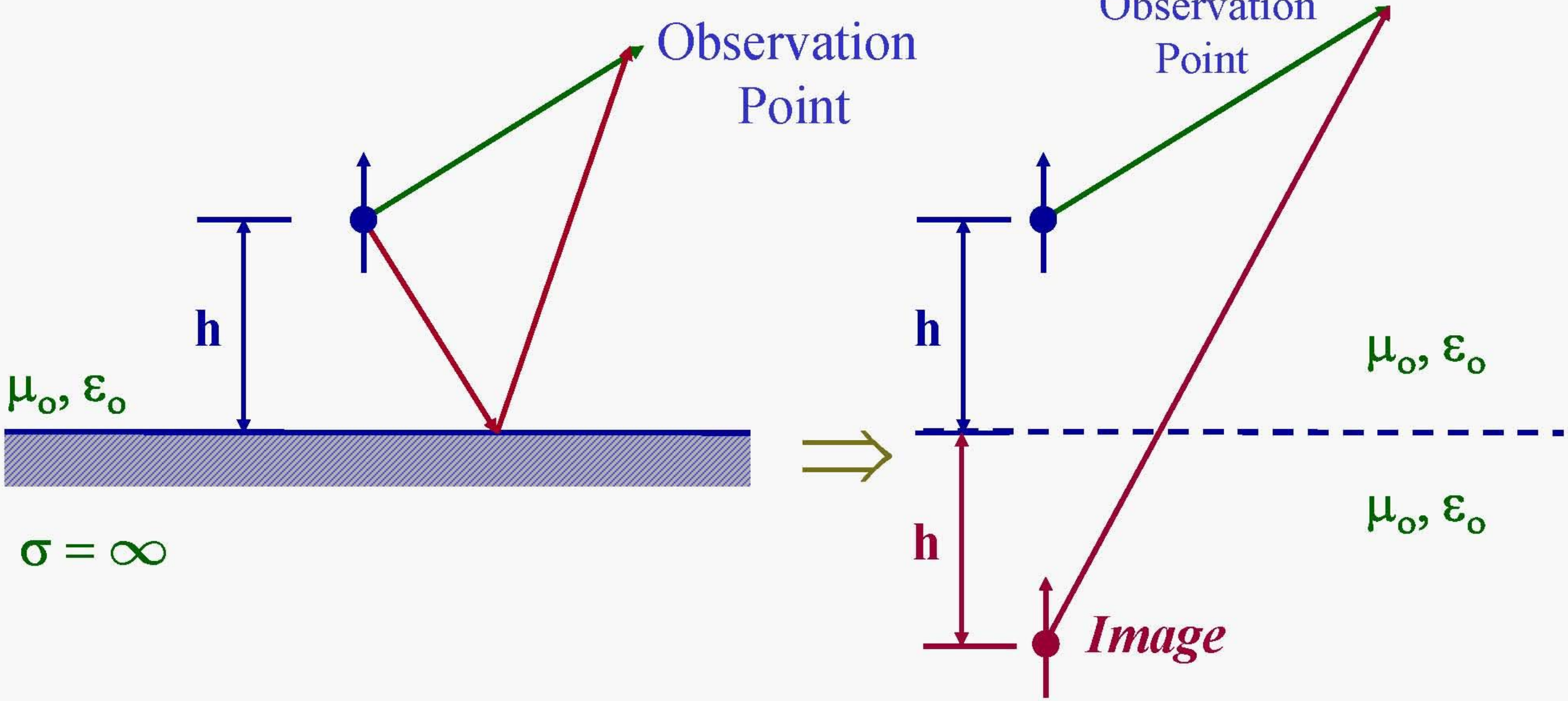


Fig. 4.12a

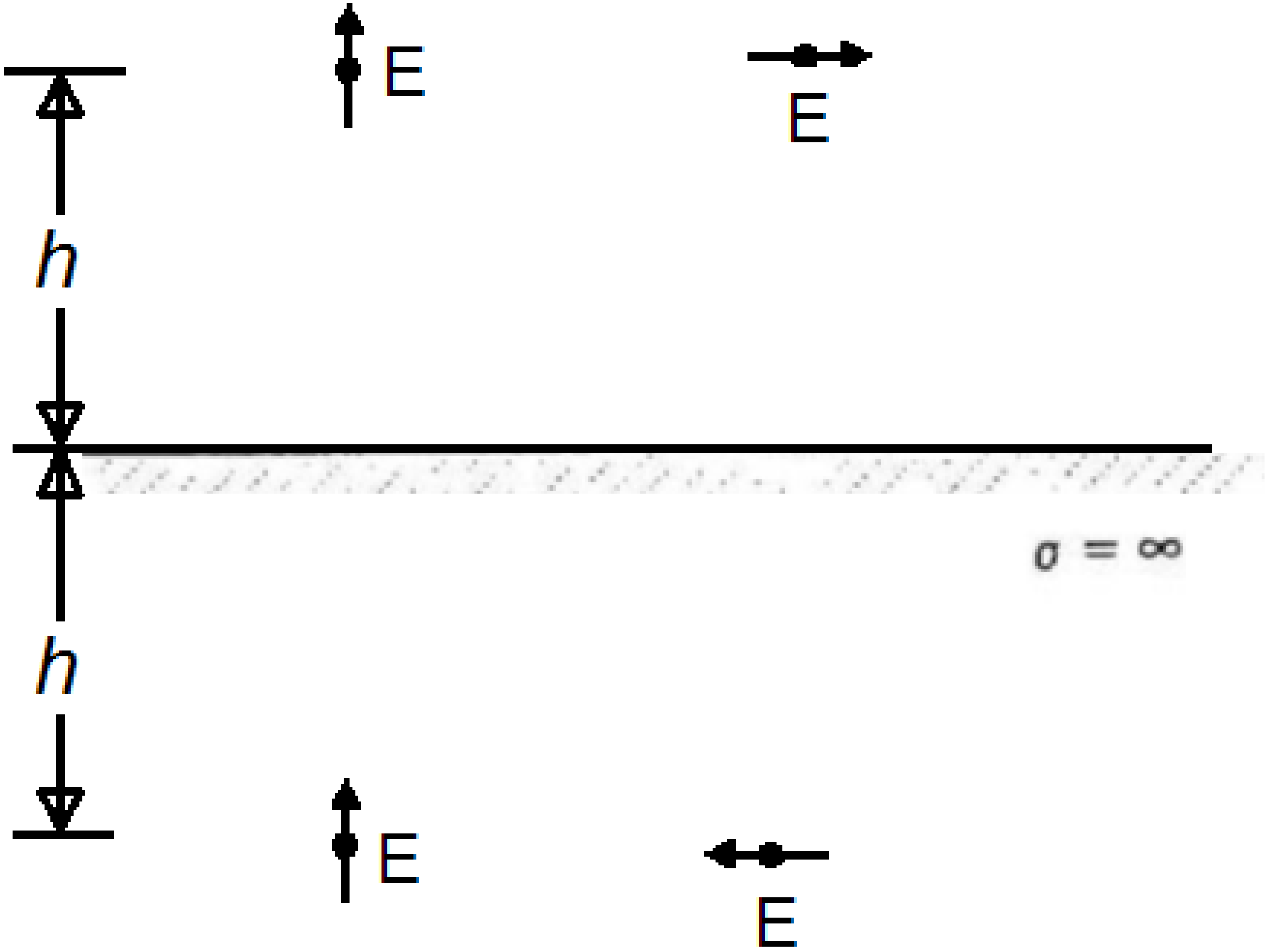
Actual and Equivalent Problems



Actual Problem

Equivalent Problem

Electric conductor



vertical dipole above infinite ground plane

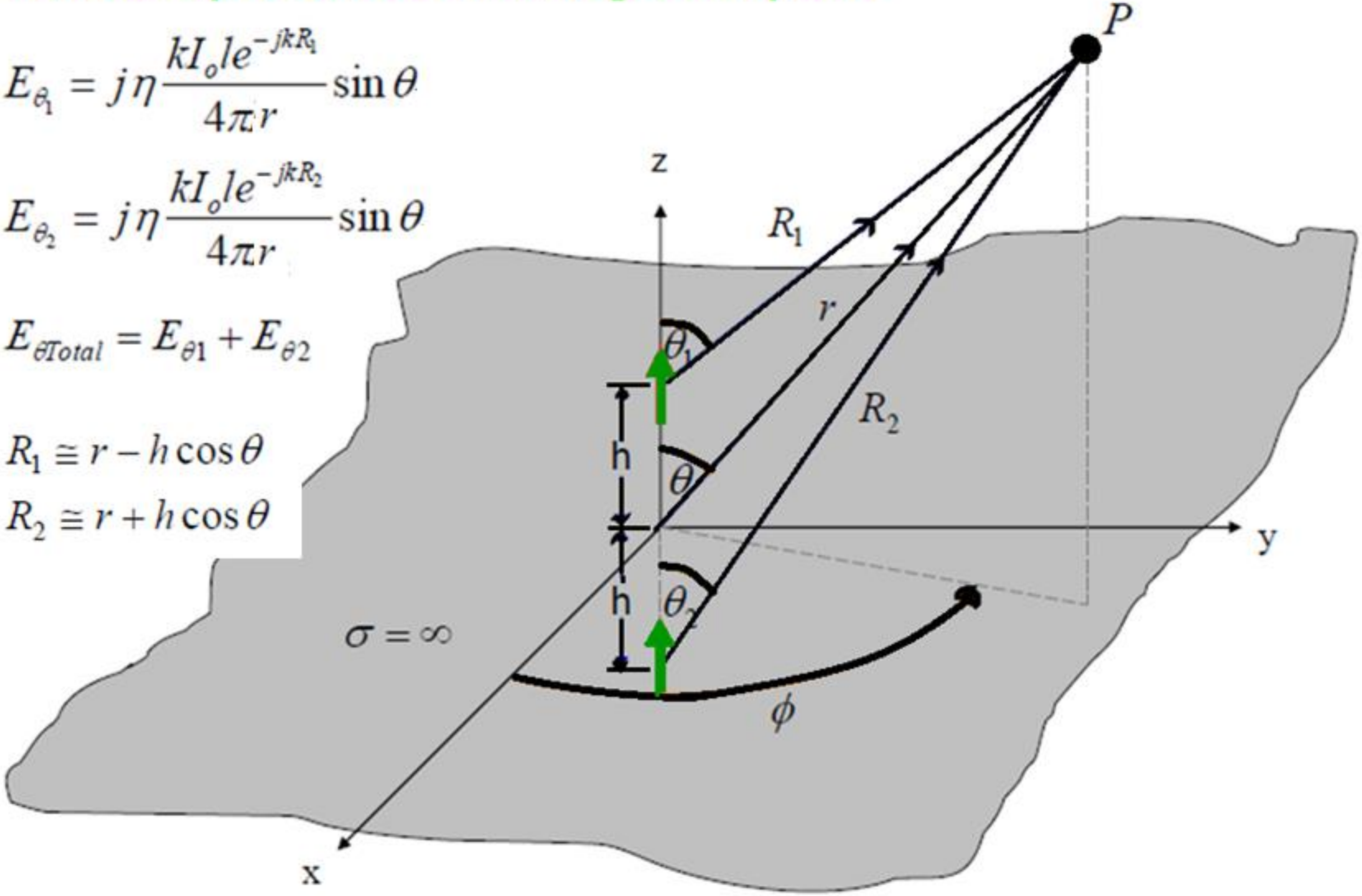
$$E_{\theta_1} = j\eta \frac{kI_o l e^{-jkR_1}}{4\pi r} \sin\theta$$

$$E_{\theta_2} = j\eta \frac{kI_o l e^{-jkR_2}}{4\pi r} \sin\theta$$

$$E_{\theta Total} = E_{\theta 1} + E_{\theta 2}$$

$$R_1 \cong r - h\cos\theta$$

$$R_2 \cong r + h\cos\theta$$



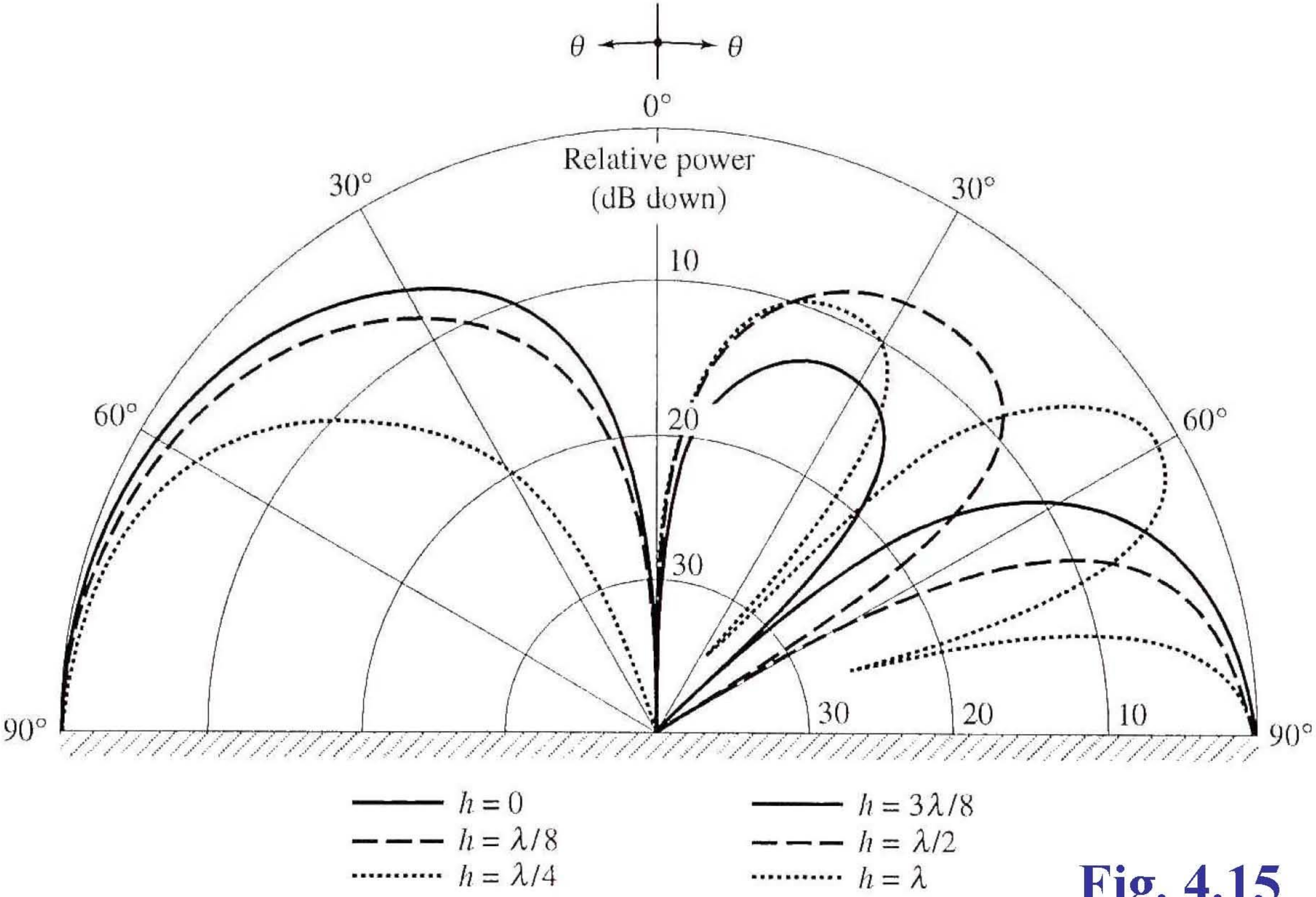
vertical dipole above infinite ground plane

$$E_\theta = j\eta \frac{kI_o l e^{-jkr_1}}{4\pi r}\sin\theta + j\eta \frac{kI_o l e^{-jkr_2}}{4\pi r}\sin\theta$$

$$E_\theta = j\eta \frac{kI_o l e^{-jkr}}{4\pi r}\sin\theta\left[e^{jkh\cos\theta} + e^{-jkh\cos\theta}\right]$$

$$E_\theta = j\eta \frac{kI_o l e^{-jkr}}{4\pi r}\sin\theta\,[2\cos(kh\cos\theta)] \qquad z \geqslant 0$$

$$E_\theta = 0 \qquad z<0$$

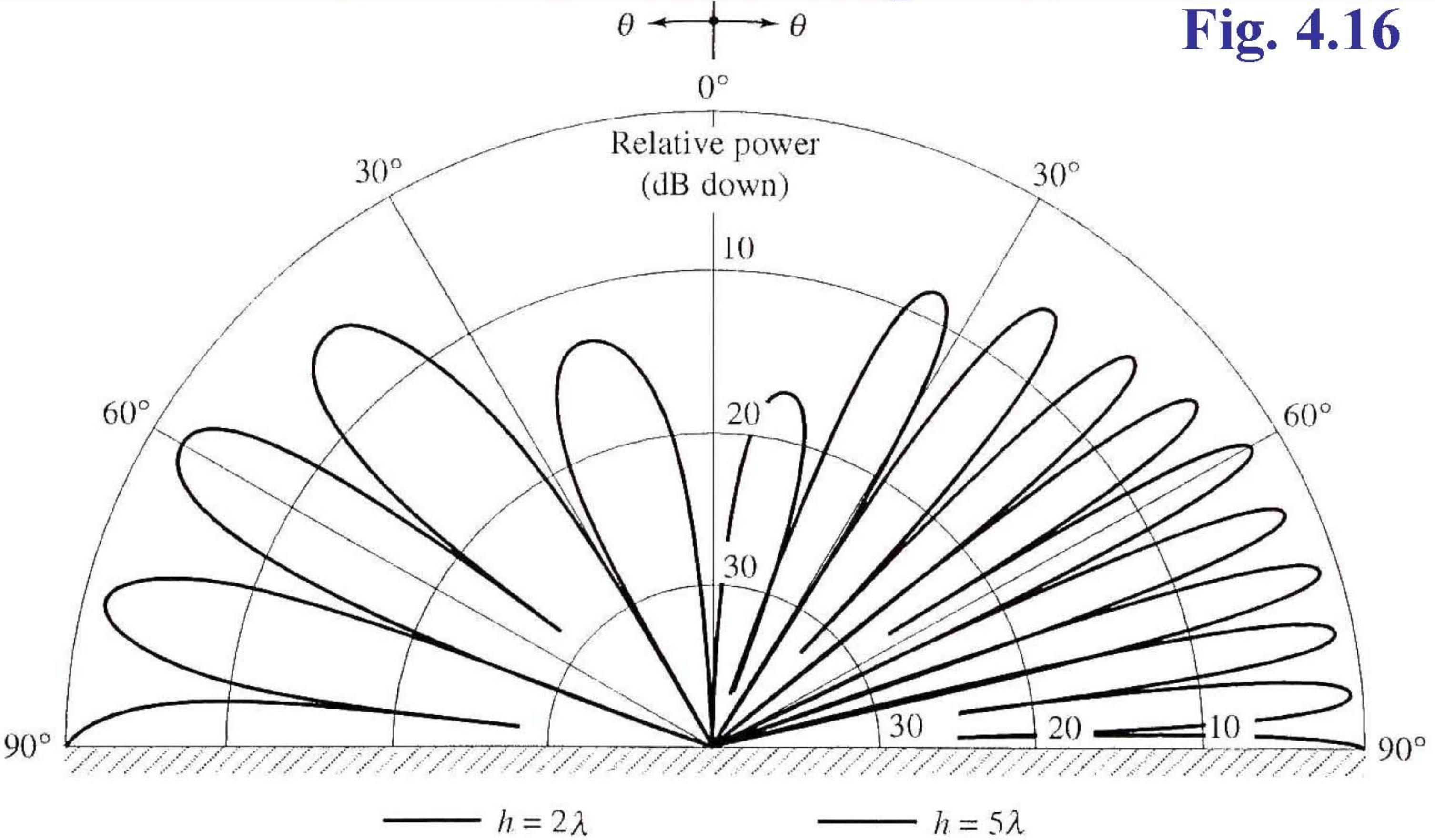


θ
θ
0°
Relative power
(dB down)
30°
30°
10
20
30
60°
60°
90°
90°
30
20
10
h = 0
h = λ/8
h = λ/4
h = 3λ/8
h = λ/2
h = λ

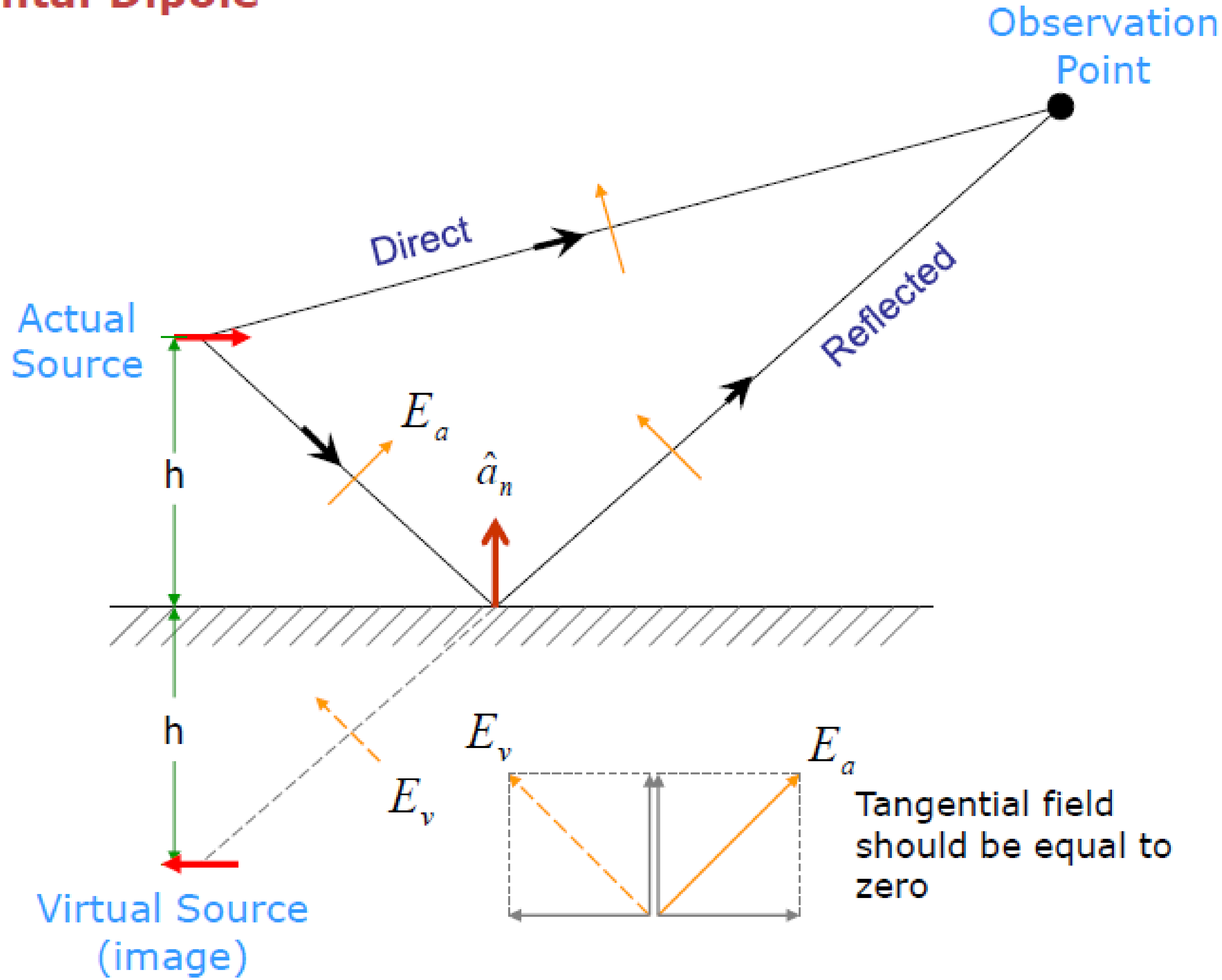
Fig. 4.15

Scalloping Of Amplitude Pattern Of Vertical Dipole

Fig. 4.16



Horizontal Dipole



Tangential field should be equal to zero

Horizontal dipole above infinite ground plane

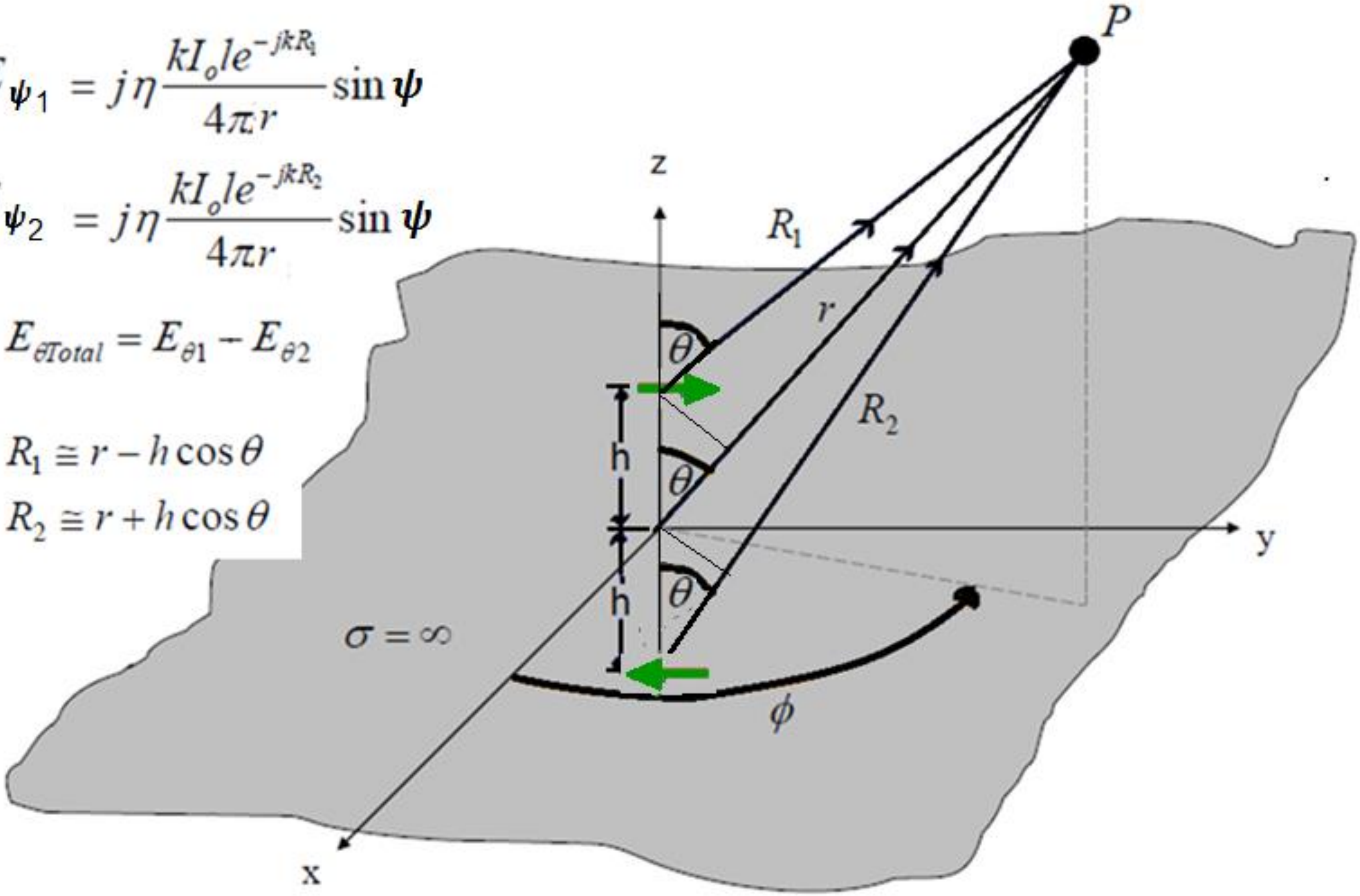
$$E_{\psi_1} = j\eta \frac{kI_o l e^{-jkR_1}}{4\pi r} \sin\psi$$

$$E_{\psi_2} = j\eta \frac{kI_o l e^{-jkR_2}}{4\pi r} \sin\psi$$

$$E_{\theta Total} = E_{\theta 1} - E_{\theta 2}$$

$$R_1 \cong r - h\cos\theta$$

$$R_2 \cong r + h\cos\theta$$



Horizontal dipole above infinite ground plane

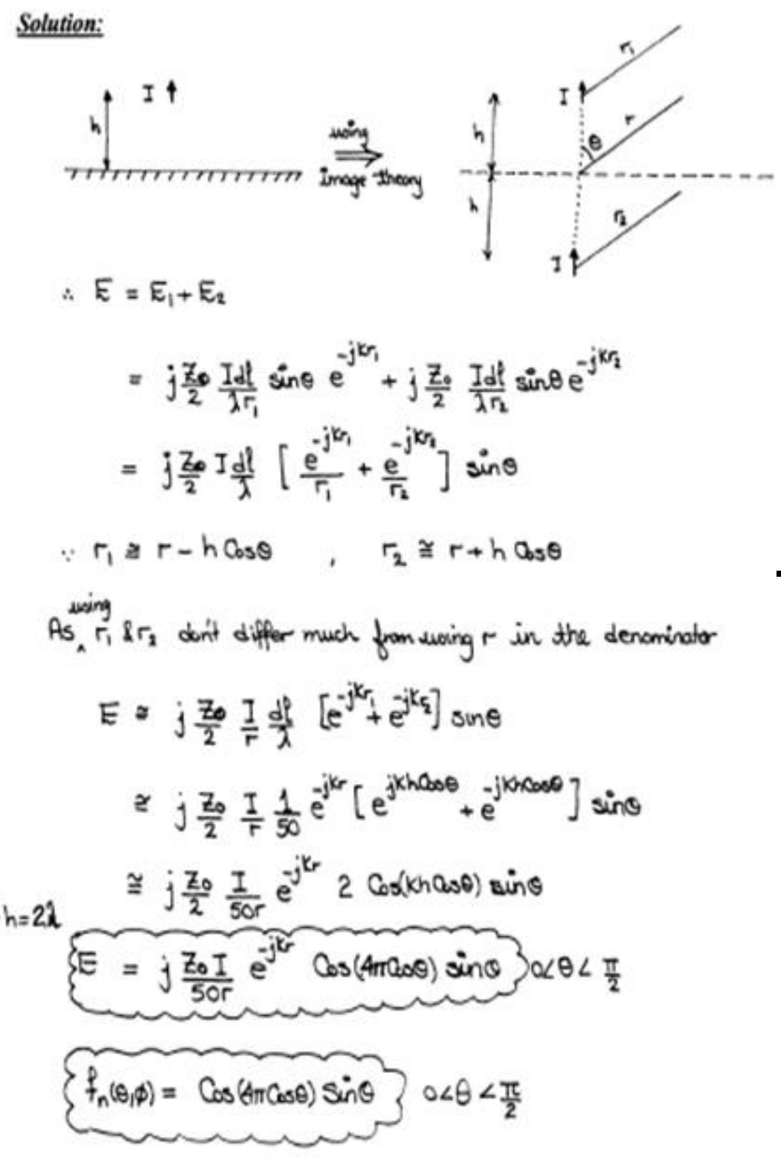
$$= j\eta \frac{kI_o le^{-jkr_1}}{4\pi r}\sqrt{1-\sin^2\theta\sin^2\phi} - j\eta \frac{kI_o le^{-jkr_2}}{4\pi r}\sqrt{1-\sin^2\theta\sin^2\phi}$$

$$E_\psi = j\eta \frac{kI_o le^{-jkr}}{4\pi r}\sqrt{1-\sin^2\theta\sin^2\phi}\left[e^{jkh\cos\theta} - e^{-jkh\cos\theta}\right]$$

$$E_\psi = j\eta \frac{kI_o le^{-jkr}}{4\pi r}\sqrt{1-\sin^2\theta\sin^2\phi}\,[2j\sin(kh\cos\theta)] \qquad z\geqslant 0$$

$$E_\psi = 0 \qquad z<0$$

An infinitesimal dipole of length l=λ/50 is placed vertically above the ground at height h=2λ Derive an equation for its field pattern



Solution:

$$\therefore\ E = E_1 + E_2$$

$$= j\frac{Z_0}{2}\frac{Idl}{\lambda r_1}\sin\theta\, e^{-jkr_1} + j\frac{Z_0}{2}\frac{Idl}{\lambda r_2}\sin\theta\, e^{-jkr_2}$$

$$= j\frac{Z_0}{2} I\frac{dl}{\lambda}\left[\frac{e^{-jkr_1}}{r_1} + \frac{e^{-jkr_2}}{r_2}\right]\sin\theta$$

$$\because\ r_1 \cong r - h\cos\theta \quad , \quad r_2 \cong r + h\cos\theta$$

As using r_1 & r_2 don't differ much from using r in the denominator

$$E \cong j\frac{Z_0}{2}\frac{I}{r}\frac{dl}{\lambda}\left[e^{-jkr_1} + e^{-jkr_2}\right]\sin\theta$$

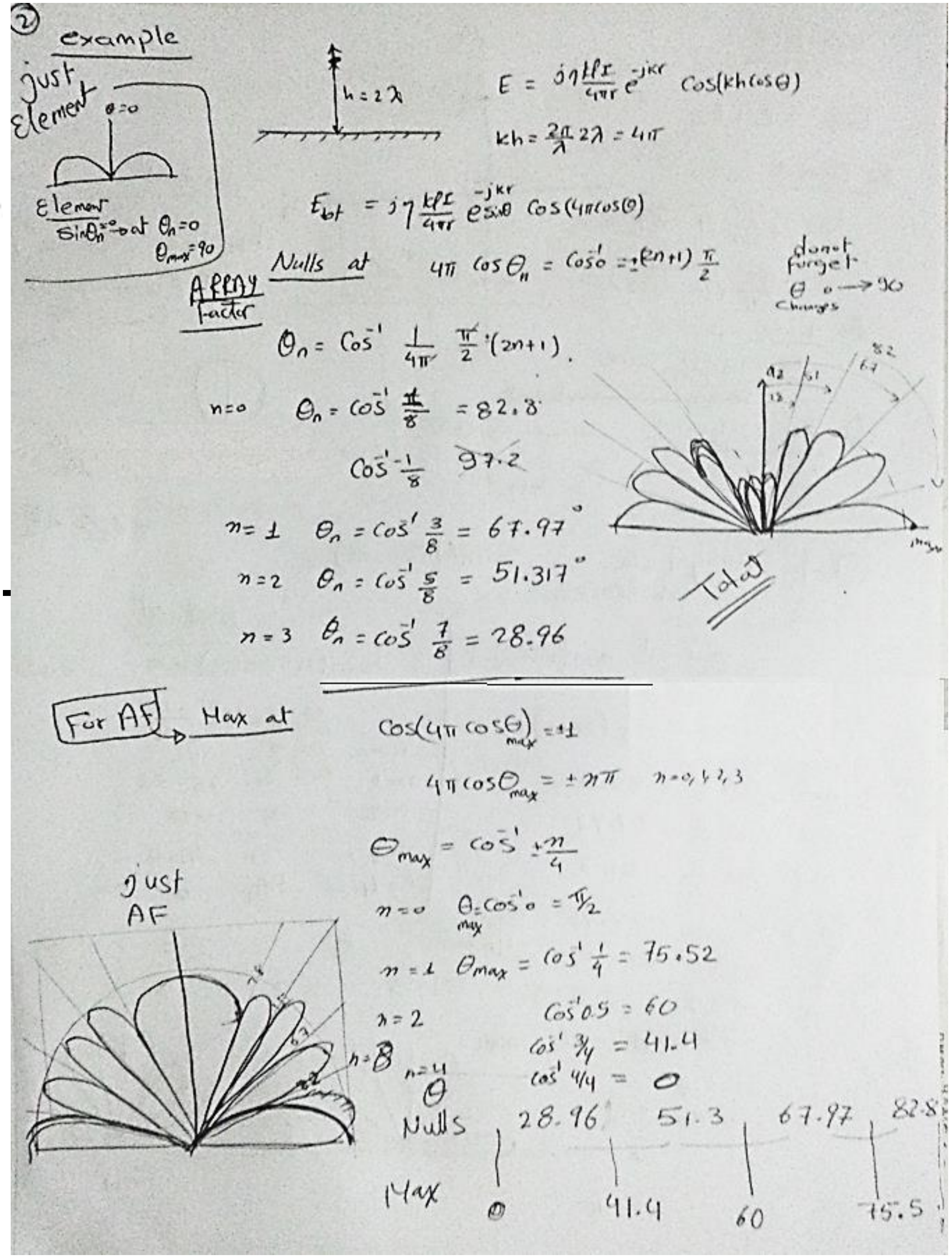
$$\cong j\frac{Z_0}{2}\frac{I}{r}\frac{1}{50}e^{-jkr}\left[e^{jkh\cos\theta} + e^{-jkh\cos\theta}\right]\sin\theta$$

$$\cong j\frac{Z_0}{2}\frac{I}{50r}e^{-jkr}\, 2\cos(kh\cos\theta)\sin\theta$$

$\because h=2\lambda$

$$\boxed{E = j\frac{Z_0 I}{50r}e^{-jkr}\cos(4\pi\cos\theta)\sin\theta} \quad 0<\theta<\frac{\pi}{2}$$

$$\boxed{f_n(\theta,\phi) = \cos(4\pi\cos\theta)\sin\theta} \quad 0<\theta<\frac{\pi}{2}$$



②

example

Element $\sin\theta_n = 0$ at $\theta_n = 0$

$\theta_{max} = 90$

$$E = j\eta\frac{k l I}{4\pi r}e^{-jkr}\cos(kh\cos\theta)$$

$$kh = \frac{2\pi}{\lambda}2\lambda = 4\pi$$

$$E_{tot} = j\eta\frac{k l I}{4\pi r}e^{-jkr}\sin\theta\cos(4\pi\cos(\theta))$$

Array Factor

Nulls at $\quad 4\pi\cos\theta_n = \cos^{-1}0 = \pm(2n+1)\frac{\pi}{2}$

don't forget θ 0 → 90 changes

$$\theta_n = \cos^{-1}\frac{1}{4\pi}\frac{\pi}{2}(2n+1).$$

n=0 $\quad \theta_n = \cos^{-1}\frac{1}{8} = 82.8$

$\cos^{-1}-\frac{1}{8}$ ~~97.2~~

n=1 $\quad \theta_n = \cos^{-1}\frac{3}{8} = 67.97°$

n=2 $\quad \theta_n = \cos^{-1}\frac{5}{8} = 51.317°$

n=3 $\quad \theta_n = \cos^{-1}\frac{7}{8} = 28.96$

For AF → Max at $\quad \cos(4\pi\cos\theta_{max}) = \pm 1$

$4\pi\cos\theta_{max} = \pm n\pi \quad n=0,1,2,3$

$$\theta_{max} = \cos^{-1}\pm\frac{n}{4}$$

n=0 $\quad \theta_{max} = \cos^{-1}0 = \pi/2$

n=1 $\quad \theta_{max} = \cos^{-1}\frac{1}{4} = 75.52$

n=2 $\quad \cos^{-1}0.5 = 60$

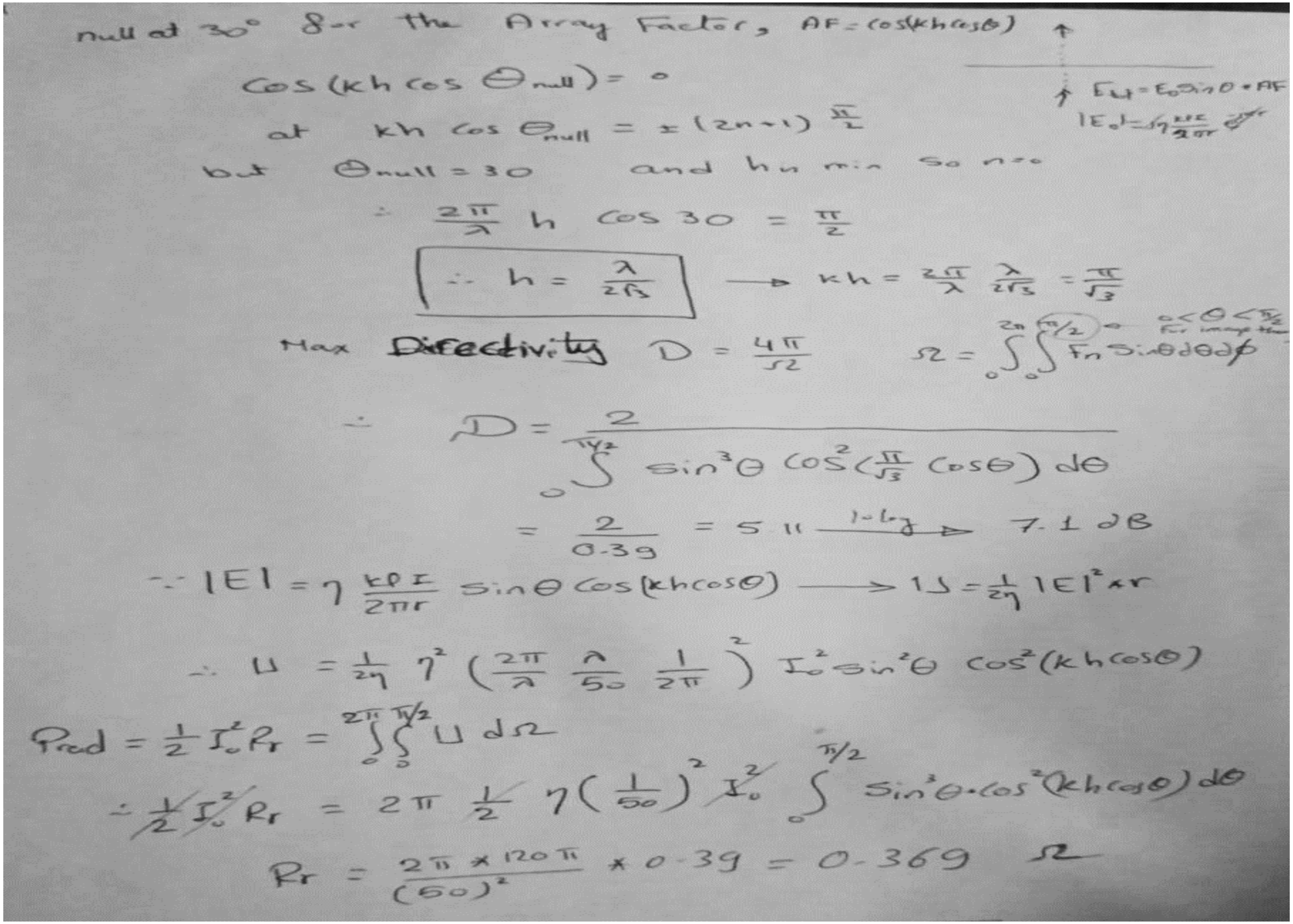
n=3 $\quad \cos^{-1}3/4 = 41.4$

n=4 $\quad \cos^{-1}4/4 = 0$

θ								
Nulls		28.96		51.3		67.97		82.8
Max	0		41.4		60		75.5	

Example (Problem 4.37)

Determine the smallest height that an infinitesimal vertical electrical dipole of l=$\lambda/50$ must be placed above an Electrical ground plane so that AF Pattern has only one null at 30º from The vertical. For that height find -The directivity -the radiation resistance



null at 30° for the Array Factor, $AF = \cos(kh\cos\theta)$

$$\cos(kh\cos\theta_{null}) = 0$$

at $kh\cos\theta_{null} = \pm(2n+1)\frac{\pi}{2}$

$E_t = E_0 \sin\theta * AF$

$|E_0| = j\eta \frac{kI\ell}{2\pi r} e^{-jkr}$

but $\theta_{null} = 30$ and h is min so $n = 0$

$$\therefore \frac{2\pi}{\lambda} h \cos 30 = \frac{\pi}{2}$$

$$\boxed{\therefore h = \frac{\lambda}{2\sqrt{3}}} \longrightarrow kh = \frac{2\pi}{\lambda}\frac{\lambda}{2\sqrt{3}} = \frac{\pi}{\sqrt{3}}$$

Max Directivity $D = \frac{4\pi}{\Omega}$

$$\Omega = \int_0^{2\pi}\int_0^{\pi/2} F_n \sin\theta \, d\theta \, d\phi$$

$0 < \theta < \pi/2$ For image theory

$$\therefore D = \frac{2}{\int_0^{\pi/2} \sin^3\theta \cos^2\left(\frac{\pi}{\sqrt{3}}\cos\theta\right) d\theta}$$

$$= \frac{2}{0.39} = 5.11 \xrightarrow{10\log} 7.1\ dB$$

$$\therefore |E| = \eta \frac{k\ell I}{2\pi r} \sin\theta \cos(kh\cos\theta) \longrightarrow U = \frac{1}{2\eta}|E|^2 * r^2$$

$$\therefore U = \frac{1}{2\eta}\eta^2\left(\frac{2\pi}{\lambda}\frac{\lambda}{50}\frac{1}{2\pi}\right)^2 I_o^2 \sin^2\theta \cos^2(kh\cos\theta)$$

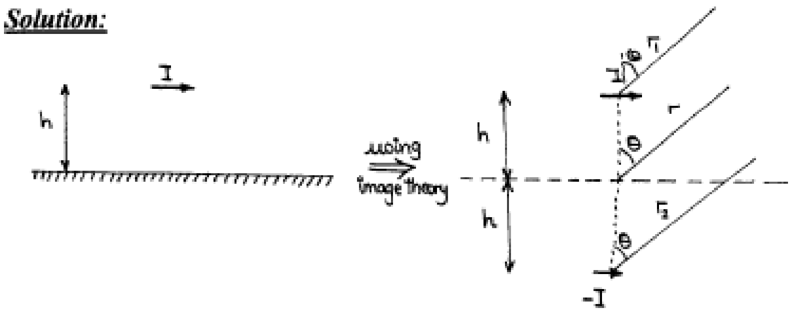
$$P_{rad} = \frac{1}{2} I_o^2 R_r = \int_0^{2\pi}\int_0^{\pi/2} U \, d\Omega$$

$$\therefore \cancel{\frac{1}{2}} \cancel{I_o^2} R_r = 2\pi \cancel{\frac{1}{2}} \eta \left(\frac{1}{50}\right)^2 \cancel{I_o^2} \int_0^{\pi/2} \sin^3\theta \cdot \cos^2(kh\cos\theta) \, d\theta$$

$$R_r = \frac{2\pi * 120\pi}{(50)^2} * 0.39 = 0.369\ \Omega$$

A doublet of length L = λ/50 is place horizontally above the ground at a height h=2λ. Derive an equation for its field pattern.

Solution:



$$\therefore \; E = E_1 + E_2$$

$$= j\frac{Z_o}{2}\frac{I\,dl}{r_1\lambda}\sin\left(\frac{\pi}{2}-\theta\right)e^{-jkr_1} - j\frac{Z_c}{2}\frac{I\,dl}{\lambda r_2}\sin\left(\frac{\pi}{2}-\theta\right)e^{-jkr_1}$$

$$= j\frac{Z_o}{2}\,I\,\frac{dl}{\lambda}\left[\frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2}\right]\operatorname{Cos}(\theta)$$

$$\because \; r_1 \cong r - h\operatorname{Cos}\theta \quad , \quad r_2 \cong r + h\operatorname{Cos}\theta$$

As in the far field zone : using r_1 & r_2 is approx. the same as using r when found in the denominator

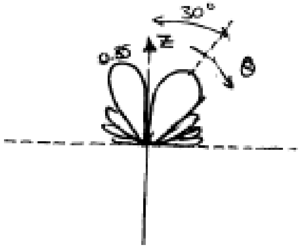
$$\therefore \; E = j\frac{Z_o}{2}\frac{I}{r}\frac{dl}{\lambda}e^{-jkr}\left[e^{jkh\operatorname{Cos}\theta} - e^{-jkh\operatorname{Cos}\theta}\right]\operatorname{Cos}\theta$$

$$= j\frac{Z_o}{2}\frac{I}{r}\frac{dl}{\lambda}e^{-jkr}\,j2\sin(kh\operatorname{Cos}\theta)\operatorname{Cos}\theta$$

$\because h = 2\lambda$

$$\boxed{\therefore \; E = -\frac{Z_o}{50}\frac{I}{r}\,e^{-jkr}\sin(4\pi\operatorname{Cos}\theta)\operatorname{Cos}\theta} \quad 0 < \theta < \frac{\pi}{2}$$

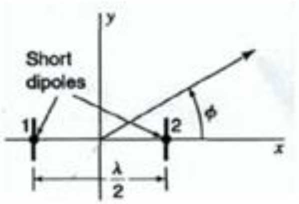
$$\boxed{\therefore \; f_n(\theta,\phi) = \sin(4\pi\operatorname{Cos}\theta)\operatorname{Cos}\theta} \quad 0 < \theta < \frac{\pi}{2}$$



principle of pattern multiplication
Two infinitesimal dipole antennas at XY Plan

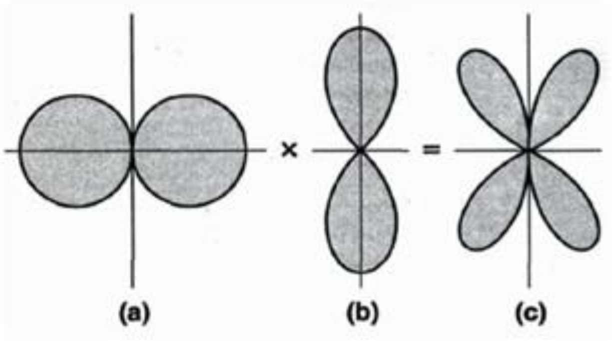
$\Theta=\pi/2$

Case(1)
Infinitesimal oriented along y direction



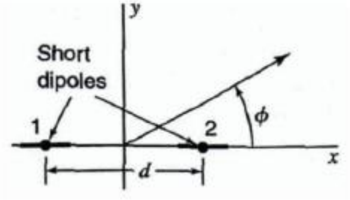
$d=\lambda/2.$

$$E = \cos\phi \cos\left(\frac{\pi}{2}\cos\phi\right)$$

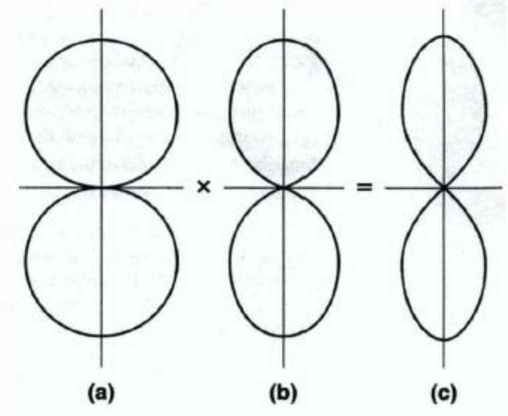


Case(2)
Infinitesimal oriented along x direction

$d=\lambda/2$



$$E = \sin\phi \cos\left(\frac{\pi}{2}\cos\phi\right)$$



LINEAR ARRAY of IDENTICAL ELEMENTS

Controls that can be used to shape the overall pattern of the antenna:

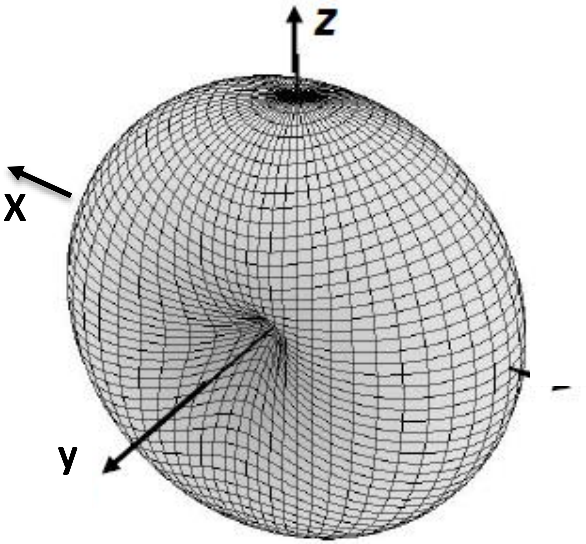
1. The geometrical configuration of the overall array (linear, circular, rectangular, spherical, etc.)

2. The relative displacement between the elements

3. The excitation amplitude of the individual elements

4. The excitation phase of the individual elements

5. The relative pattern of the individual elements



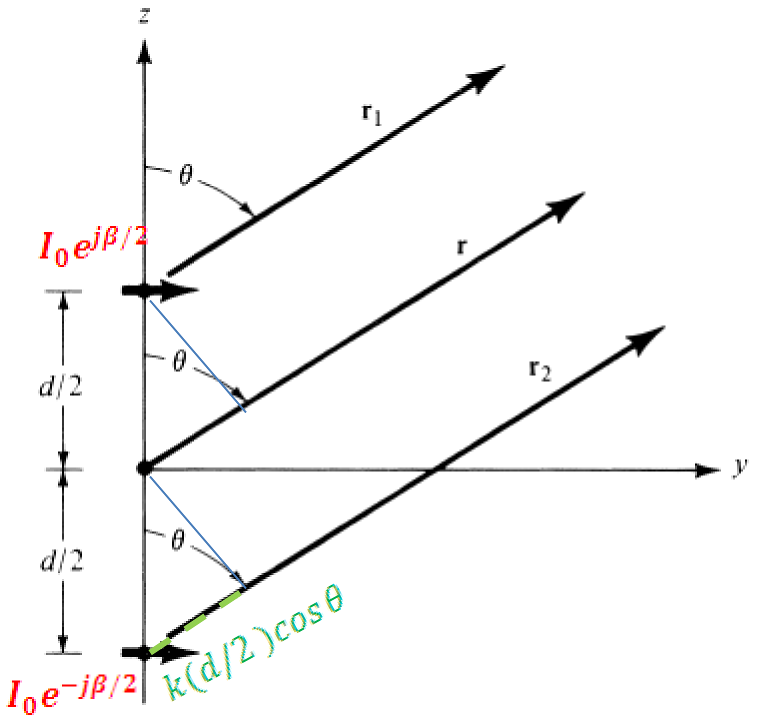
Dipole oriented to y axis

TWO-ELEMENT ARRAY

For two infinitesimal horizontal dipoles positioned along the *z*-axis Oriented at y axis and

suppose we study tot E at YZ plane

$$\mathbf{E}_t = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{a}}_\theta j\eta \frac{kI_0 l}{4\pi}\left\{\frac{e^{-j[kr_1-(\beta/2)]}}{r_1}\cos\theta_1 + \frac{e^{-j[kr_2+(\beta/2)]}}{r_2}\cos\theta_2\right\}$$



(b) Far-field observations

$$E_t = \hat{a}_\theta j\eta \frac{kI_0 l e^{-jkr}}{4\pi r}\cos\theta[e^{+j(kd\cos\theta+\beta)/2} + e^{-j(kd\cos\theta+\beta)/2}]$$

$$E_t = \hat{a}_\theta j\eta \frac{kI_0 l e^{-jkr}}{4\pi r}\cos\theta\left\{2\cos\left[\frac{1}{2}(kd\cos\theta+\beta)\right]\right\}$$

$$\boxed{\text{E(total)} = [\text{E(single element at reference point)}] \times [\text{array factor}]}$$

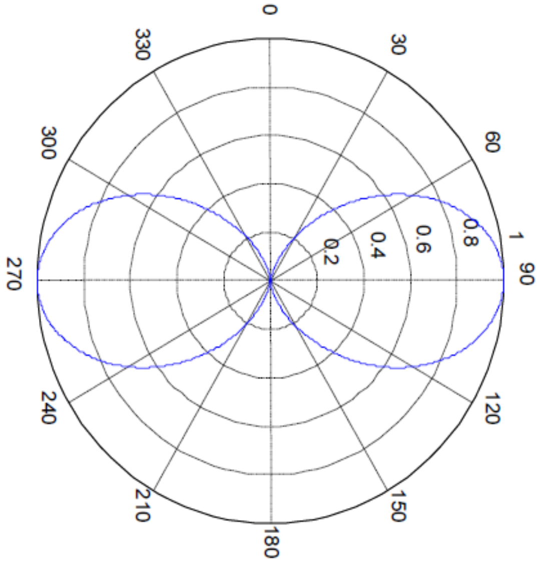
Arrays of two isotropic point sources (ARRAY FACTOR PATTERN)

Case 1 same amplitude and phase (*β=0*) and for d=λ/2 Kd/2=π/2 antennas on az axis

(AF)n= $\cos[(kd/2)\cos\theta]$**=** $\cos[(\pi/2)\cos\theta]$

Max at $\frac{\pi}{2}\cos\theta_m = m\pi$, $m = 0,1,2,\ldots$ $\theta_m = \cos^{-1}(0) = \frac{\pi}{2}, -\frac{\pi}{2}$

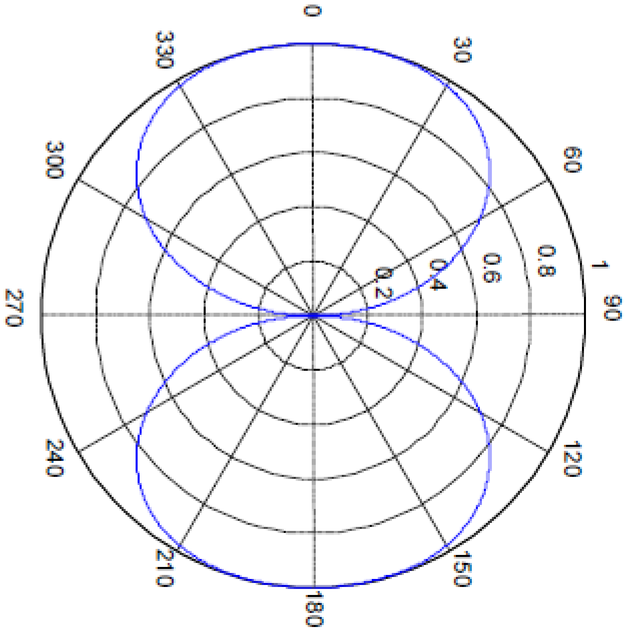
Nulls at $\frac{\pi}{2}\cos\theta_n = \pm(2m+1)\pi/2$ $\theta_n = \cos^{-1}(\pm 1)$**=0,π**



- **Case 2 same amplitude and opposite phase (β=180) and for d=λ/2 Kd/2=π/2**

$$\left|(AF)_n\right| = \cos\left(\frac{kd}{2}\cos\theta + \pi/2\right) = \sin\left(\frac{kd}{2}\cos\theta\right)$$

θ	0	10	20	30	40	50	60	70	80	90
AF_n	1	.999	.995	.978	.933	.847	.707	.5	.269	0

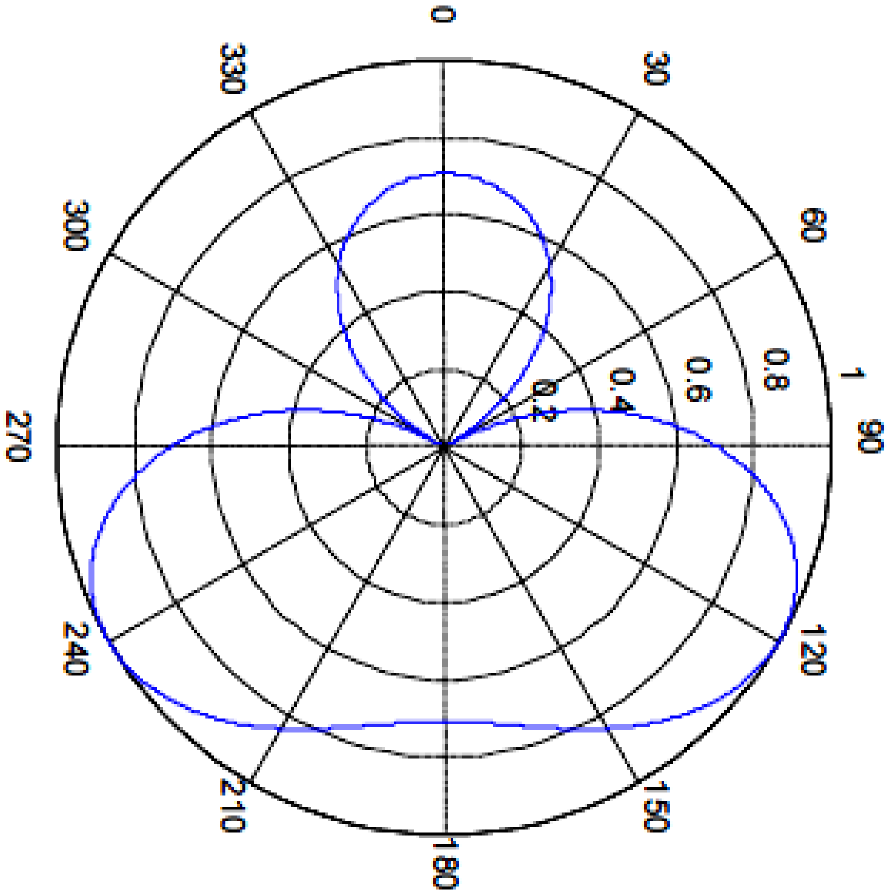


Changing phase of source currents shift to 180 change AF pattern as seen in Fig.

- Case 3 (a) same amplitude and quadrature phase (β/2=π/4) and for d=λ/2
Kd/2=π/2

$$\left|(AF)_n\right| = \cos(\frac{\pi}{2}\cos\theta + \pi/4)$$

θ	0	10	20	30	40	50	60	70	80
$\|AF_n\|$	.707	.69	.637	.543	.406	.22	0	.245	.49
θ	90	100	110	120	130	140	150	160	170
$\|AF_n\|$	.707	.87	.969	1	.975	.913	.839	.77	.723
θ	180								
$\|AF_n\|$	.707								

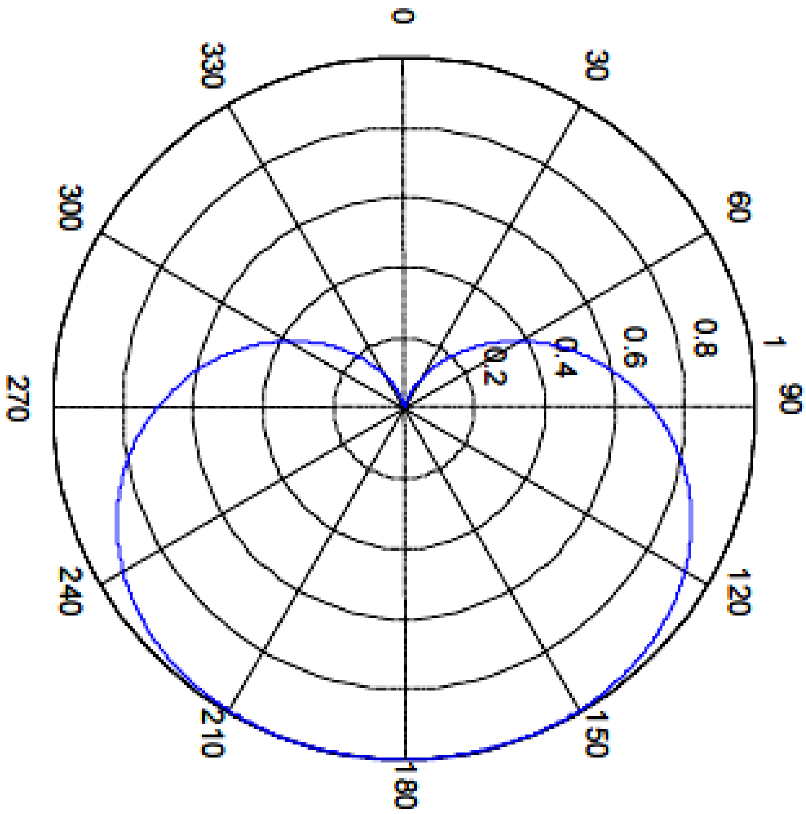


Most radiation directed toward lower half

- **Case 3 (b) same amplitude and quadrature phase ($\beta/2=\pi/4$) and for $d=\lambda/4$ $Kd/2=\pi/4$**

Will be described by phase accumulation

$$\left|(AF)_n\right| = \cos\left(\frac{\pi}{4}\cos\theta + \pi/4\right)$$



Can we explain max occurred at 180 and nulls at 0 in terms of phase accumulation of phases Generated by array elements at observation point, …this illustrated at next slide.

Pattern Synthesis